A Linearized Analytical Solution Of PDE Heat Transfer In A Solid Material Excited By Light With Radiant Thermal Losses

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ABSTRACT

A method to linearize partial differential equations (PDEs) of heat diffusion and its initial with boundary conditions in a wide interval of temperature is presented. Thus, an analytical solution of the heat conduction problem was achieved in a virtual experiment performed in a lead wall with an ideally established configuration. For other arrangements where an analytical solution is not possible, simple

standard programs can be used as an alternative to solve the linearized partial differential equations obtained with the proposed method. A sophisticated computer program as the standard gold method of COMSOL Multiphysics® was used for the evaluation of error of procedure. It was validated that the relative error of the analytical solution, in comparison to the digital simulation with COMSOL Multiphysics, can be found without the need to own and use the COMSOL Multiphysics program. For our configuration, the error is 1.16 % and focuses on the initial part of the process.

Keywords: Heat transfer, PDE, Radiant losses, Analytical solution.

1 INTRODUCTION

In order to find the thermal properties of a material, it is mandatory to conduct thermal experiments whose theoretical foundation are the partial differential equations (PDEs) of heat transfer [1]. In many cases, when working on a wide interval of temperature, these PDEs are nonlinear because the thermal properties of the materials and the heat transfer coefficient h [$\text{Wm}^{-2}\text{K}^{-1}$] are dependent on the temperature. Besides, the heat loss coefficient involves the sum of three components, as shown in Eq. 1:

$$h = h_{conv} + h_{cond} + h_{rad} \quad (1)$$

where h_{conv} , h_{cond} and h_{rad} represent heat exchange with the environment: the heat loss due to thermal convection, thermal conduction, and thermal radiation, respectively. It is known that h_{conv} and h_{cond} components are significative affected by the conditions of the experiment such as the construction of the measurement arrangement [2-4]; the air velocity in the measurement laboratory [5]; the type of air movement - laminar or turbulent [6]; the emissivity of the sample surface [7]; the roughness of the sample [8-9], the local temperature of the sample [10-11], etc. Such conditions make the experimental measurement, prediction, and theoretical description of both components be very hard and even impractical. However, h_{rad} part can be expressed with the Stefan-Boltzmann Law [12] and using short calculations the Eq. 2 can be obtained:

$$h_{rad,exact} = \varepsilon\sigma(T^2 + T_0^2)(T + T_0) \quad (2)$$

where T [K] is the experimental temperature, T_0 [K] is the environment temperature, σ is the Stefan-Boltzmann constant, $\sigma = 5.670373 \times 10^{-8}$ [$\text{Wm}^{-2}\text{K}^{-4}$], ε represents the thermal emissivity of the sample. For accurate and reliable results, it is preferable to work under a vacuum where it can be assumed that $h_{conv} = h_{cond} = 0$ [13-16]. Such parameters can be considered constant when the temperature

difference, $T - T_0$, is small compared to the ambient temperature, T_0 [17-19]. However, h_{rad} can be approximated by expansion in the Taylor series, with the Eq. 3:

$$h_{rad,approx} = 4\varepsilon\sigma T_0^3 \quad (3)$$

To be able to use the approximation indicated by Eq. 3, it is necessary to work in a narrow interval of temperatures around the environment temperature. However, it has been noticed that when the experimental temperature T significative increases over T_0 the errors provoked by the use of these approximations rapidly become unacceptable. That is shown in Fig. 1 where the relative error calculated with Eq. 4 as a function of the relative change of temperature $(T - T_0)/T_0$ was plotted:

$$Relative\ error\ [\%] = \frac{h_{rad,exact} - h_{rad,aprox}}{h_{rad,exact}} \times 100 \quad (4)$$

From this perspective, in case of a big difference, $T - T_0$, the use of the approximation given by Eq. 3 is not permitted.

Sophisticated programs such as the program COMSOL Mutiphysics® are generally used as the gold method to resolve in a precise way the nonlinear PDEs [20]. Therefore, in this paper is presented a simplified approach to linearize the PDE mainly related to heat diffusion. The objective of this paper is to find an analytical solution to the heat conduction problem in order to achieve similar results to those obtained with the standard gold method of COMSOL Mutiphysics®, which uses nonlinear PDEs. At the same time, the variation of the thermal parameters of the material and heat loss coefficient with the temperature will be taken into account. A virtual experiment will be done to prove the usefulness of the proposed method as well as to determine the discrepancy values with the exact solution.

Figure 1. Relative error as a function of the relative change of temperature

2 DESCRIPTION OF THE VIRTUAL EXPERIMENT

The virtual experimental arrangement consists of a lead wall with two infinite dimensions and with finite, but thermally immense, thickness, with flat interfaces in the vacuum, as schematically presented in Fig. 2.

Concerning Fig. 2, it was assumed that (a) heat transfer takes place in one direction in regards to the thickness. (b) There was only radiation energy loss in both walls ($x = 0$ and $x = d$). (c) The chosen thickness of the wall was so big, $d = 10$ m, to guarantee that in the rear face, heat losses were relatively minor. Thus, for $x = d$, a heat loss coefficient value of approximately zero was considered. (d) Both interfaces were painted with optically diffused black paint (model 359 black, COMEX) [21], to ensure a thermal emissivity $\varepsilon = 0.96$ [22]. (e) The interface in $x = 0$ was heated by a uniform and constant light after its start, with intensity I_0 [W/m^2], coming from the left zone of the vacuum. (f) At the moment $t = 0$, the light was turned on, before $t = 0$, the light was deliberately off. (g) The experimental temperature interval chosen for the left interface of the lead wall was relatively wide: minimum temperature $T_1 = 300$ K, maximum temperature $T_2 = 600$ K. A maximum simulation time of 6000 s was used for the virtual experiment. Evidently, in such cases, the approximation is given by Eq. 3 cannot be used.

Based on the assumptions above, the heat transfer in the lead wall was only governed by conduction phenomena described by Eq. 5:

$$\rho C_p \frac{\partial T}{\partial t} = \nabla(k \nabla T) \quad (5)$$

for unidimensional heat conduction, Eq. 5 was reduced to the following spatial-temporal PDE:

$$\rho C_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left(k \frac{\partial T}{\partial x} \right) \quad (6)$$

subjected to the following nonlinear boundary and initial conditions:

$$-k \frac{\partial T}{\partial x} = \sigma A \varepsilon (T_0^4 - T^4) + q_0 \quad \text{in } x = 0 \quad (7)$$

$$-k \frac{\partial T}{\partial x} = \sigma A \varepsilon (T^4 - T_0^4) \quad \text{in } x = d \quad (8)$$

$$T = T_0 \quad \text{for } t = 0 \quad (9)$$

where q_0 represents the energy flux contributed by the incident light. From the above boundary conditions (Eq. 7 to 9), it can be appreciated that the effect of heat transfer by convection was not included because both sides of the wall were exposed to the vacuum. It can also be noticed that as a consequence of the nonlinearity of the boundary conditions and the dependence of the thermal parameters of the metal on the temperature, there is not a direct analytical solution of Eq. 6.

3 THEORY TO FIND AN ANALYTICAL SOLUTION

Among the main thermal properties of the lead wall to find the analytical solution was: ρ - the mass density of the material [kg/m^3], c_p - the specific heat capacity [$\text{J}/(\text{kg} \cdot \text{K})$], k - the thermal conductivity [$\text{W}/(\text{m} \cdot \text{K})$], h_{rad} - the radiation heat loss coefficient [$\text{W}/(\text{m}^2 \cdot \text{K})$], all of them as a function of temperature [20].

To achieve an analytical solution of the heat diffusion PDE, it is necessary to obtain constant values of the thermal properties ρ , c_p , k and h_{rad} . Therefore, it is proposed to calculate their statistical average values in the range of working temperatures, T_1 and T_2 , using mathematical functions for $\rho(T)$, $c_p(T)$ y $k(T)$ from the literature [20] according to Eq. 10 to 13:

$$h_{rad}^{stat} = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} h_{rad}(T) dT \quad (10)$$

$$= \frac{\varepsilon \sigma}{T_2 - T_1} \left(\frac{1}{4} T^4 + \frac{1}{3} T_0 * T^3 + \frac{1}{2} T_0^2 * T^2 + T_0^3 * T \right)_{T_1}^{T_2}$$

$$\rho^{stat} = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \rho(T) * dT \quad (11)$$

$$c_p^{stat} = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} c_p(T) * dT \quad (12)$$

$$k^{stat} = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} k(T) * dT \quad (13)$$

Working temperatures, T_1 and T_2 , will give, in the case of the absence of the program COMSOL Multiphysics, the limits of the analytical solution and will allow to evaluate the maximum error of the calculations.

The PDE for unidimensional heat conduction (Eq. 6), as well as the boundary and initial conditions, are expressed in their respective linearized form as:

$$\frac{\partial T}{\partial t} = \frac{k}{\rho C_p} \frac{\partial^2 T}{\partial x^2} \quad (14)$$

$$-k \frac{\partial T}{\partial x} = h_{rad}(T_0 - T) + q_0 \quad \text{in } x = 0 \quad (15)$$

$$-k \frac{\partial T}{\partial x} = h_{rad}(T - T_0) \quad \text{in } x = d \quad (16)$$

$$T = T_0 \quad \text{for } t = 0 \quad (17)$$

Consequently, heat transfer in the lead wall was solved analytically through the non-trivial solution procedure that involved the following steps:

1.- Obtain dimensionless equations. With the following definitions:

$$U = T - T_0, \quad X = \frac{x}{d}, \quad N_{Bi} = \frac{h_{rad} \cdot d}{k}, \quad \tau = \frac{t}{d^2}, \quad Q_0 = \frac{q_0}{h_{rad}}, \quad \alpha = \frac{k}{\rho C_p} \quad (18)$$

the transformed version of the heat conduction problem becomes:

$$\frac{\partial U}{\partial \tau} = \alpha \frac{\partial^2 U}{\partial X^2} \quad (19)$$

where:

$$\frac{\partial U}{\partial X} = N_{Bi}(U - Q_0) \quad \text{in } X = 0 \quad (20)$$

$$\frac{\partial U}{\partial X} = -N_{Bi}U \quad \text{in } X = 1 \quad (21)$$

$$U = 0 \quad \text{for } \tau = 0 \quad (22)$$

2.- Application of the principle of superposition. In this case, the following definition was used

$$U = U_s(X) + \hat{U}(\tau, X) \quad (23)$$

where U_s and $\hat{U}$ are the solution in steady-state and the contribution of the transient state, respectively. The two following problems were obtained:

Problem 1

$$\frac{\partial^2 U_s}{\partial X^2} = 0 \quad (24)$$

Subject to

$$\frac{\partial U_s}{\partial X} = N_{Bi}(U_s - Q_0) \quad \text{in } X = 0 \quad (25)$$

$$\frac{\partial U_s}{\partial X} = -N_{Bi}U_s \quad \text{in } X = 1 \quad (26)$$

Problem 2

$$\frac{\partial \hat{U}}{\partial \tau} = \alpha \frac{\partial^2 \hat{U}}{\partial X^2} \quad (27)$$

Subject to

$$\frac{\partial \hat{U}}{\partial X} = N_{Bi}\hat{U} \quad \text{in } X = 0 \quad (28)$$

$$\frac{\partial \hat{U}}{\partial X} = -N_{Bi}\hat{U} \quad \text{in } X = 1 \quad (29)$$

$$\hat{U} = -U_s(X) \quad \text{for } \tau = 0 \quad (30)$$

3.- Resolution of the problems 1 and 2 through the separation of variables:

The solution for the ordinary differential equation (ODE) of **Problem 1** was:

$$U_s(X) = C_1X + C_2 \quad (31)$$

and by applying its corresponding boundary conditions, it was obtained:

$$U_s(X) = \frac{-Q_0}{2 + N_{Bi}} (N_{Bi}X - 1 - N_{Bi}) \quad (32)$$

The solution for the PDE of **Problem 2** was found as:

$$\hat{U} = \sum_{n=1}^{\infty} P_n \left(\cos(\lambda_n X) + \frac{N_{Bi}}{\lambda_n} \sin(\lambda_n X) \right) e^{-\lambda_n^2 \alpha \tau} \quad (33)$$

whereas the variable λ_n was obtained from the solution set of the characteristic equation:

$$2N_{Bi} \left(\frac{\lambda}{\lambda^2 - N_{Bi}^2} \right) - \tan \lambda = 0 \quad (35)$$

Finally, based on the information of the phenomenon, the solution can be obtained by applying the following algorithm:

- Calculate the parameter:

$$N_{Bi} = \frac{h_{rad} * d}{k} \quad (36)$$
- Get all characteristic values from:

$$2N_{Bi} \left(\frac{\lambda_n}{\lambda_n^2 - N_{Bi}^2} \right) - \tan(\lambda_n) = 0 \quad \forall n = 1, 2, 3, \dots, \infty \quad (37)$$

- Calculate the parameters:

$$X = \frac{x}{d}; \quad \tau = \frac{t}{d^2}; \quad Q_0 = \frac{q_0}{h_{rad}}; \quad \alpha = \frac{k}{\rho C_p} \quad (38)$$

- Calculate the steady solution:

$$U_s(X) = \frac{Q_0}{2 + N_{Bi}} (1 + N_{Bi} - N_{Bi}X) \quad (39)$$

- Calculate the truncated transient solution in a sufficient number of terms, n_u :

$$P_n = \frac{-2Q_0}{2 + N_{Bi}} \frac{(\lambda_n^2 - N_{Bi}^2) \sin \lambda_n + 2N_{Bi} \lambda_n (1 - \cos \lambda_n) + N_{Bi}^2 \lambda_n}{(\lambda_n^2 - N_{Bi}^2) \cos \lambda_n \sin \lambda_n + 2N_{Bi} \lambda_n (1 - \cos^2 \lambda_n) + N_{Bi}^2 \lambda_n + \lambda_n^3} \quad (40)$$

$$\hat{U}(\tau, X) = \sum_{n=1}^{n_u} P_n \left(\cos(\lambda_n X) + \frac{N_{Bi}}{\lambda_n} \sin(\lambda_n X) \right) e^{-\lambda_n^2 \alpha \tau} \quad (41)$$

- Rebuild the complete solution from Eq. 23 and including U definition of Eq. 18:

$$T(\tau, X) = T_0 + U_s(X) + \hat{U}(\tau, X) \quad (42)$$

In the virtual experiment, the following parameters were used:

$T_0 = T_1 = 300 \text{ K}$	$T_2 = 600 \text{ K}$
$d = 10 \text{ m}$	$\rho = 11218 \text{ kg/m}^3$
$C_p = 136.42 \text{ J/(kg K)}$	$k = 33.3745 \text{ W/(m K)}$
$h_{rad} = 13.141 \text{ W/(m}^2 \text{ K)}$	$I_0 = 30000 \text{ W/m}^2$

It is crucial to notice that Eq. 10 to 13 with temperatures values of $T_1 = T_0 = 300 \text{ K}$ and $T_2 = 600 \text{ K}$ were applied to determine the values of the mass density, specific heat capacity, thermal conductivity, and of the radiation heat loss coefficient.

On the other hand, the following composed parameters were calculated:

$$N_{Bi} = h * d / k = 3.937467$$

$$Q_0 = q_0 / h_{rad} = 2282.914 \text{ K}$$

$$\alpha = k / (\rho * C_p) = 2.18\text{E-}05 \text{ m}^2/\text{s}$$

and also the first 58 characteristic values, obtained from the solution of Eq. 27, were considered and are presented in the following table:

Table 1. The first 58 characteristic values obtained with Eq. 37

n	λ_n	n	λ_n	n	λ_n
1	2.1443	21	62.9568	41	125.7263
2	4.5650	22	66.0925	42	128.8664
3	7.2753	23	69.2287	43	132.0065
4	10.1640	24	72.3653	44	135.1467
5	13.1483	25	75.5024	45	138.2870
6	16.1852	26	78.6399	46	141.4273
7	19.2530	27	81.7776	47	144.5677
8	22.3401	28	84.9157	48	147.7082
9	25.4399	29	88.0540	49	150.8486
10	28.5484	30	91.1925	50	153.9892
11	31.6634	31	94.3312	51	157.1297
12	34.7830	32	97.4701	52	160.2704
13	37.9061	33	100.6092	53	163.4110
14	41.0320	34	103.7484	54	166.5517
15	44.1602	35	106.8878	55	169.6924
16	47.2900	36	110.0273	56	172.8332
17	50.4213	37	113.1669	57	175.9739
18	53.5539	38	116.3066	58	179.1147
19	56.6874	39	119.4464		
20	59.8217	40	122.5863		

Therefore, given the temperature interval between $T_0 = T_1 = 300$ K and $T_2 = 600$ K to the left interface of the lead wall, temperatures at different locations at any time within the lead wall can be now determined.

4 DIGITAL RESULTS

The results of the analytical solution of the EDP, $T_{analytical}(x, t)$, as resolved in section “Theory to find an analytical solution”, are presented graphically below. These results are compared with those digital results obtained by simulation in COMSOL Multiphysics, $T_{COMSOL}(x, t)$. The discrepancy between the analytical solution and COMSOL Multiphysics digital simulation was quantified based on the normalized root-mean-square error (*NRMSE*) [23], which is defined as:

$$NRMSE [\%] = \sqrt{\frac{1}{m} \frac{\sum_{i=1}^m (T_{i,analytical} - T_{i,COMSOL})^2}{\sum_{i=1}^m (T_{i,COMSOL})^2}} \times 100 \quad (43)$$

where m is the number of the data points for one graph, $T_{i,analytical}$ and $T_{i,COMSOL}$ are the analytical solutions and COMSOL Multiphysics solution values, respectively, corresponding to the i -th data point ($i = 1 \dots m$).

Figure 3. The value of *NRMSE* (Eq. 43) vs. number n_u of members of the sum

A study of the behavior of the *SSE* regarding the number of members used in the partial sum of Eq. 41 is presented in Fig. 3. The transient contribution of the analytical solution (Eq. 33) is given as the sum of infinite numbers of the members. To do quantitative calculations, it is necessary to sum a finite number n_u of members (Eq. 41), and in this case, calculations were done with numbers of the members from $n_u = 1$ to $n_u = 58$, which are shown in Table 1. From Fig. 3 it can be observed that *NRMSE* decreased as the number of the members was increased, and this became practically stabilized for $n_u = 57$ and 58 values where the difference of the value of *NRMSE* was smaller than 0.005 %.

Figure 4. Spatio-temporal behavior of $T(x,t)$ by two different methods: a) The analytical solution $T_{\text{analytical}}(x,t)$ for $n_u=58$, and b) The COMSOL Multiphysics digital simulation $T_{\text{COMSOL}}(x,t)$.

Figure 4-a shows the response surface plot of the temperature as a function of thickness and time generated with the analytical solution, $T_{analytical}(x, t)$, with $n_u = 58$ members of the sum (Eq. 41). For comparison, the corresponding response surface plot of the temperature distribution generated with COMSOL Multiphysics digital simulation, $T_{COMSOL}(x, t)$, is presented in Fig. 4-b:

It was well established that temperature gradients are significant in the layer 2 m in thickness close to the surface undergoing illumination. That means that reasonable uniform temperature distribution with values similar to T_0 was observed in the domain $2 \leq x \leq 10$ m. Therefore, it was decided to perform the calculations for $NRMSE$ (Eq. 43) and to construct the previous figures only in an interval of x between $0 < x < 2$ m.

5 EVALUATION OF ERRORS OF THE ANALYTICAL SIMULATION

Suppose the calculations are made using the analog model, with the set of main thermal parameters of the virtual experiment ($\rho(T)$ - the mass density of the material [kg/m^3], $c_p(T)$ - the specific heat capacity [$\text{J}/(\text{kg}\cdot\text{K})$], $k(T)$ - the thermal conductivity [$\text{W}/(\text{m}\cdot\text{K})$], $h(T)$ - the heat loss coefficient [$\text{W}/(\text{m}^2\cdot\text{K})$]) with its values for extreme temperatures of $T_1 = 300$ K and $T_2 = 600$ K, respectively, $T_0 = 300$ K. Let their corresponding solutions be $T_1^{analogic}(x, t)$ and $T_2^{analogic}(x, t)$, respectively. The first part of the hypothesis is that the analogic solution of the PDEs of heat diffusion using the statistical values of the main thermal parameters (Eq. 10 to 13), written down as $T_{stat}^{analogic}(x, t)$, for all the values of x and of t , satisfies the inequality:

$$T_1^{analogic}(x, t) > T_{stat}^{analogic}(x, t) > T_2^{analogic}(x, t) \quad (44)$$

The simulation in COMSOL Multiphysics is done using the thermal properties values corresponding to the same temperatures, will give results $T_1^{COMSOL}(x, t)$ and $T_2^{COMSOL}(x, t)$, respectively. The second part of the hypothesis is that the solution of the simulation with COMSOL Multiphysics of the PDEs of heat diffusion, written down as $T^{COMSOL}(x, t)$, for all the values of x and of t , satisfies the inequality:

$$T_1^{COMSOL}(x, t) > T^{COMSOL}(x, t) > T_2^{COMSOL}(x, t) \quad (45)$$

The calculations demonstrate that the hypothesis is accomplished in both parts. To be able to present the results $T_1^{analogic}(x, t)$, $T_{stat}^{analogic}(x, t)$ and $T_2^{analogic}(x, t)$ of the analytical solution in a 2D graph and at the same time not to lose the generality, the temperature is averaged by x , which is:

$$T_{averaged\ by\ x}(t) = \frac{1}{2} \int_{x=0}^{x=2} T(x,t) dx \quad (46)$$

In such a case, the following graph (Fig. 5-a) is available for the analytical solution, where “Aver300A” is the temperature averaged by x for the analytical solution with thermal parameters for 300 K (Aver = averaged). “AverAn” is the temperature averaged by x for the analytical solution with thermal statistical parameters. “Aver600A” is the temperature averaged by x for the analytical solution with thermal parameters for 600 K. As expected, the temperatures for the thermal properties for 300 K are higher. The temperatures for the thermal properties for 600 K are lower than the three curves at all times, which is, inequality is accomplished (Eq. 44).

Figure 5. The temperatures averaged by x calculated by a) analytical solution and b) COMSOL Multiphysics simulation. Curves with square points corresponding to T_1 , triangle points corresponding to T_2 , and circle points corresponding to a) statistical values (Eq. 10 to 13) and b) non-linear solution with COMSOL.

For the simulation with COMSOL Multiphysics, results are shown in Fig. 5-b, where “Aver300C” is the temperature averaged by x for the simulation COMSOL Multiphysics with thermal parameters for 300 K. “AverCOM” is the temperature averaged by x for the simulation COMSOL Multiphysics. “Aver600C” is the temperature averaged by x for the simulation COMSOL Multiphysics with thermal parameters for 600 K. It can be seen that inequality is accomplished (Eq. 45).

Placing the two graphs of Fig. 5-a, and Fig. 5-b, in a coordinates system, we see that there is a satisfactory coincidence (Fig. 6):

Figure 6. Comparison of the temperatures averaged by x of the analytical solution and the simulation with the program COMSOL Multiphysics of the Fig. 5-a, and Fig. 5-b. The averaged temperatures are shown as follows: “Aver300A” – black squares; “AverAn” – red circles; “Aver600A” – blue triangles; “Aver300C” – green triangles; “AverCOM” – pink triangles; “Aver600C” – brown triangles

The maximum relative discrepancy for the three pairs of the curves is smaller than 0.1 %. That allows the interval $[T_{600K}(x, t); T_{300K}(x, t)]$ to be used as representative of the analytical solution error. Comparing the relative difference DR between the two surfaces ($0 < x < 2$ m, $0 < t < 6000$ s): first $T_1^{analogic}(x, t)$ and $T_2^{analogic}(x, t)$ and then $T_1^{COMSOL}(x, t)$ and $T_2^{COMSOL}(x, t)$. The calculation of DR is done by the following equations:

$$DR_{analytical} [\%] = \sqrt{\frac{1}{m} \sum_{i=1}^m \left(\frac{T_2^{analogic}(x, t) - T_1^{analogic}(x, t)}{T_2^{analogic}(x, t) + T_1^{analogic}(x, t)} \right)^2} \times 100 \quad (47)$$

and

$$DR_{COMSOL} [\%] = \sqrt{\frac{1}{m} \sum_{i=1}^m \left(\frac{T_2^{COMSOL}(x, t) - T_1^{COMSOL}(x, t)}{T_2^{COMSOL}(x, t) + T_1^{COMSOL}(x, t)} \right)^2} \times 100 \quad (48)$$

where m is the number of the data points for one graph or surface, corresponding to the i -th data point ($i = 1 \dots m$). For the indicated surfaces the relative differences DR show the values: $DR_{analytical} = 2.26 \%$ and $DR_{COMSOL} = 2.23 \%$, equal considering.

Figure 7. The relative difference $DR_{analytical}$ calculated according to Eq. 47 (red dots) and the relative difference DR_{COMSOL} of the Eq. 48 for the data of the simulation with COMSOL Multiphysics (black dots).

To see if there is a more in-depth match between $DR_{analytical}$ and DR_{COMSOL} , the values of these two parameters for a row of the matrix $T(x, t)$ are calculated with the scan time t ($0 < t < 6000$ s) and with the family parameter of x curve values. The results are presented in Fig. 7.

An excellent concordance between both curves is seen. That allows the possibility of exchanging the data for the relative difference between both cases – COMSOL Multiphysics and the analytical solution, and it explains the practical equality between $DR_{analytical}$ and DR_{COMSOL} for the surfaces $T(x, t)$.

Figure 8. $NRMSE$ vs. x for scanning time t from 0 until 6000 s.

The value of $NRMSE$ among the data of the simulation of COMSOL Multiphysics – our “golden solution,” and the results of the analytical solution for all the surface $T(x, t)$ is (Eq. 43): $NRMSE = 1.16$ %.

Figure 8 shows the calculated values of $NRMSE$ for each row of the matrix $T(x, t)$ making a scan at time t ($0 < t < 6000$ s) and with the family parameter of x curve values.

Figure 9 shows a comparison of the curves of the previous parameters with the standardized $NRMSE$. Standardization is done with respect to the quotient:

$$C = DR_{analytical}/NRMSE = 2.26 \% / 1.16 \% = 1.9482 \approx 2 \quad (4)$$

The similarity between the curve of $DR_{analytical}$ and the standardized $NRMSE$ curve is noticeable; the relative error between these two curves reaches the value (using the Eq. 43): $NRMSE(\text{between the two curves}) = 3.40 \%$, that is a relatively small error.

Subsequently, it can be presented the value of $NRMSE$ between COMSOL Multiphysics and analytical solution: $NRMSE \approx DR_{analytical}/2$. The value of $DR_{analytical}$ can be found without the necessity to use COMSOL Multiphysics.

Therefore it is recommended the following work procedure:

1. The wall material and temperatures of interest are chosen: environment temperature T_0 , minimum temperature T_1 and maximum temperature T_2 .
2. The values of the main thermal properties are looked for: $\rho(T)$ - the mass density of the material [kg/m^3], $c_p(T)$ - the specific heat capacity [$\text{J}/(\text{kg}\cdot\text{K})$], $k(T)$ - the thermal conductivity [$\text{W}/(\text{m}\cdot\text{K})$] as a function of temperature T within the interval $[T_1, T_2]$. That could be done based on the corresponding charts in the literature.

Figure 9. The $DR_{analytical}$ (red dots), DR_{COMSOL} (black dots), $NRMSE$ (blue dots) and $NRMSE$ standardized by the coefficient C (Eq. 49, green dots) vs. x .

3. Calculations are done for ρ^{stat} , c_p^{stat} , k^{stat} , h^{stat} .
4. Heat diffusion equations are analytically resolved with these parameters. The solution is obtained as $T_{stat}^{analogic}(x, t)$ within the interval $[T_1, T_2]$.
5. With the values of these thermal properties: ρ , c_p , k , h , for the temperatures T_1 and T_2 , respectively, analytical solutions are looked for: $T_1^{analogic}(x, t)$ and $T_2^{analogic}(x, t)$, respectively. Using the Eq. 47 the $DR_{analytical}$ discrepancy between the two solutions is sought. In such case the approximate value of $NRMSE$ between the result of the digital simulation with COMSOL Multiphysics (our “golden method”) $T^{COMSOL}(x, t)$ and the analytical solution $T_{stat}^{analogic}(x, t)$, is obtained dividing $DR_{analytical}$ by 2.

Of course, this procedure, with other materials and other configurations, must be verified.

6 DISCUSSION

A method to linearize heat diffusion PDEs with their boundary and initial conditions in a wide interval of temperatures is presented. One of the advantages of this linearization method is that it is possible, in many experimental configurations, to look for the analytical solution of the problem. Even if for other experimental configurations, this is not possible, for the digital solution of linearized heat diffusion PDEs simple programs can be used to resolve linear PDEs, which are part of traditional software: Mathematica [24], PTC Mathcad [25], EXCEL [26], etc. Another advantage is that the relative error between the analytical solution and the digital simulation with COMSOL Multiphysics can be found without the necessity to own and use the program COMSOL Multiphysics. It is noticed that a big part of the discrepancy between the analytical solution and simulation COMSOL Multiphysics is concentrated closer to the initial moment $t = 0$.

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