Interactions of the solutions of a coupled system of Benjamin-Bona-Mahony type in weighted spaces

DOI: 10.46932/sfjdv2n1-058

Received in: November 1st, 2020

Accepted in: December 30th, 2020

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ABSTRACT

Interactions of the solutions are a part of collisions of wave systems. We prove the existence of the solutions of a coupled system of Benjamin- Bona-Mahony (BBM) type in a weighted Sobolev space with initial data. we are concerned with a dispersive system for an appropriate class of nonlinearities. The study for a coupled system of BBM type without weighted was initiated by Bisognin, V. [3] Further results can be found in Bona, J.-Ponce, G.-Saut, J.C. [4] .

Keywords: global existence, coupled system, weighted Sobolev.

1 INTRODUCTION

Recently Chen and Wang [1] studied the stability of the solitary wave solutions to a coupled BBM system

$$u_t - u_{xxt} + u_x + P(u, v)_x = 0$$

$$v_t - v_{xxt} + v_x + Q(u, v)_x = 0$$

$$\text{In } -\infty < x < \infty, t > 0 \quad (0.0)$$

We introduce the coupled system of Benjamin- Bona-Mahony type in a weighted Sobolev:

$$\begin{aligned} u_t - u_{xxt} - av_{xxt} + a_1 u_x + a_2 v^p v_x + u^p u_x + a_3 (u^p v)_x &= 0 \\ v_t - v_{xxt} - au_{xxt} + a_1 v_x + a_2 u^p u_x + v^p v_x + a_3 (uv^p)_x &= 0 \\ \text{In } -\infty < x < \infty, t > 0 \end{aligned} \quad (1.1)$$

Here a, a_1, a_2 and a_3 are constants. Assume now that $a_1 > 0, 0 < a < 1$ and $p \in \mathbb{N}$ with $p > 1$; the unknowns are u, v defined in $\mathbb{R}^{2+} = (-\infty, \infty) \times [0, \infty)$. We consider the initial conditions

$$u(x, 0) = u_0(x) \text{ and } v(x, 0) = v_0(x) \quad (1.2)$$

for the coupled system (1.1). Let us first introduce some notation. Let $X_r = H_r^2(\mathbb{R}) \times H_r^2(\mathbb{R})$. Where $H_r^2(\mathbb{R})$ is a Hilbert space with the inner product $(u, v)_{r,2} = (M_r \Lambda^2 u, M_r \Lambda^2 v)_2$. Here $(\cdot, \cdot)_2$ is the inner product in $L^2(\mathbb{R})$. For $r \in \mathbb{R}^+$, let Λ^2 be the Pseudo-differential operator defined by $\Lambda^2 = (I - \partial_x^2)$. Furthermore, we set $M_r f(x) = (1 + |x|^2)^{r/2} f(x)$. In order to apply semigroup theory, we must first transform the coupled system (1.1) to a first-order system. For this purpose, We set

$$\begin{aligned} A &= \begin{pmatrix} \Lambda^2 & -a\partial_x^2 \\ -a\partial_x^2 & \Lambda^2 \end{pmatrix}, B = \begin{pmatrix} a_1 & 0 \\ 0 & a_1 \end{pmatrix}, w = \begin{pmatrix} u \\ v \end{pmatrix} \\ (Fw)_x &= \begin{pmatrix} \left(a_2 \frac{v^{p+1}}{p+1} + \frac{u^{p+1}}{p+1} + a_3 (u^p v)_x \right)_x \\ \left(a_2 \frac{u^{p+1}}{p+1} + \frac{v^{p+1}}{p+1} + a_3 (v^p u)_x \right)_x \end{pmatrix}, \text{ and we rewrite (1.1) and (1.2) as} \\ \begin{cases} Aw_t + B w_x + (Fw)_x = 0, & x \in \mathbb{R}, t > 0 \\ w(x, 0) = w_0(x), & x \in \mathbb{R} \end{cases} \quad (1.3) \\ \text{Where } w_0 &= \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}. \end{aligned}$$

1.1 PRELIMINARIES

Let X a Hilbert space and $m \in \mathbb{N}$, $T > 0$ we denote by $C^m([0, T]; X)$ the space of vector. Valued functions $u: [0, T] \rightarrow X$ m -times continuously differentiable in $[0, T]$ with the usual norm:

$$\mu(x) = (1 + |x|^2)^{1/2}$$

For every $s \in \mathbb{R}$, the Sobolev spaces $H^s(\mathbb{R})$ are introduced as usual with the norm

$\|u\|_s = |\mu^s \hat{u}|_2$ where the $|\cdot|$ denotes the standard norm of $L^2(\mathbb{R})$ and $\hat{u}$ is the Fourier transform of u . We also denote by

$$\Lambda^s = (I - \partial_x^2)^{s/2}$$

The Bessel potential of order s .

If $r \in \mathbb{R}$ we denote by M_r the multiplication operator with the function $\mu^r(x)$. While if $r, s \in \mathbb{R}$ we denote by $H_r^s(\mathbb{R}) = H^s(\mathbb{R}, \mu^r(x)dx)$ the completion of Schwarz space $\mathcal{S}(\mathbb{R})$ in the norm

$$|u|_{r,s} = |M_r \Lambda^s u|_2$$

$H_r^s(\mathbb{R})$ is a Hilbert space with the inner product

$(u, v)_{r,s} = (M_r \Lambda^s u, M_r \Lambda^s v)_2$ and its dual is defined by $(H_r^s(\mathbb{R}))' = H_{-r}^{-s}(\mathbb{R})$. When $r = 0$, the space H_0^s coincide with the usual norm Sobolev space. The case $r = s = 0$ we put $H_0^0(\mathbb{R}) = L^2(\mathbb{R})$.

We introduce the Hilbert space $\mathcal{S}_r^s(\mathbb{R})$ defined by

$$\mathcal{S}_r^s(\mathbb{R}) = H_r^0(\mathbb{R}) \cap H_0^s(\mathbb{R})$$

For every $r, s \in \mathbb{R}$ with the inner product

$$(((u, v)))_{r,s} = (u, v)_{r,0} + (u, v)_{0,s}$$

We give some properties

i) For every $r_1, s_1, r_2, s_2 \in \mathbb{R}$ and $\theta \in (0,1)$

$$[H_{r_1}^{s_1}, H_{r_2}^{s_2}]_{\theta} = H_{(1-\theta)r_1 + \theta r_2}^{(1-\theta)s_1 + \theta s_2}$$

Where $[X, Y]_{\theta}$ is the interpolation space of X and Y .

ii) For every $r, s \in \mathbb{R}$ there exists c_1, c_2 positive constants such that

$$c_1 |u|_{r,s} \leq |\mathcal{F}^{-1}(M_s \mathcal{F}(M_r u))|_2 \leq c_2 |u|_{r,s}$$

For every $s \in H_r^s(\mathbb{R})$

iii) For every $r, s \in \mathbb{R}$, $\mathcal{F}(H_r^s(\mathbb{R})) = H_s^r(\mathbb{R})$

iv) If $r_1 \geq r_2, s_1 \geq s_2$ then

$$H_{r_1}^{s_1} \subset H_{r_2}^{s_2}, \quad \mathcal{S}_{r_1}^{s_1} \subset \mathcal{S}_{r_2}^{s_2} \quad \text{with continuous injection.}$$

v) For every $r, s \in \mathbb{R}$ and $\theta \in (0,1)$

$$\mathcal{S}_r^s(\mathbb{R}) \subset [H_r^0(\mathbb{R}), H_0^s(\mathbb{R})]_\theta = H_{(1-\theta)r}^{\theta s}$$

vi) If $r_1, s_1, r_2, s_2, r, s \in \mathbb{R}$ and $u \in H_{r_1}^{s_1}(\mathbb{R})$, $v \in H_{r_2}^{s_2}(\mathbb{R})$ then $uv \in H_r^s(\mathbb{R})$ whenever

$$s_1, s_2 \geq s, \quad s_1 + s_2 - s \geq 1/2$$

Moreover there exists a constant $c > 0$ such that:

$$|uv|_{r,s} \leq c|u|_{r_1,s_1}|v|_{r_2,s_2}$$

For some other details one find in (cf.[4],[14]).

2 RESULTS

2.1 Global existence and mild solution

Theorem 1. Let $r \geq 0$ if $u_0 \in H^4(\mathbb{R}) \cap H_r^2(\mathbb{R})$

then the initial value problem

$$\begin{cases} \partial_t u + \partial_x u + u \partial_x u - \partial_x^2 \partial_t u = 0, & x \in \mathbb{R}, t > 0 \\ u(x, 0) = u_0(x), & x \in \mathbb{R} \end{cases} \quad (2)$$

Has a unique solution

$$u \in C^0([0, \infty); H^4(\mathbb{R}) \cap H_r^2(\mathbb{R})) \cap C^1([0, \infty); H_r^2(\mathbb{R}))$$

Before proceeding to the proof, we establish some preliminary results. We know that (2) is equivalent to the following initial-value problem:

$$\begin{cases} \partial_t u = Bu + Bfu, & x \in \mathbb{R}, t > 0 \\ u(x, 0) = u_0(x), & x \in \mathbb{R} \end{cases}$$

Where $B = -\Lambda^{-2} \partial_x$, $f(s) = s^2/2$

Lemma1. $B: H_r^2(\mathbb{R}) \rightarrow H_r^2(\mathbb{R})$ is a linear bounded operator.

Proof. In fact, for $u \in H_r^2(\mathbb{R})$ we have

$$\begin{aligned} |Bu|_{r,2}^2 &= |M_r \Lambda^2 \Lambda^{-2} \partial_y u|_2^2 = |M_r \partial_y u|_2^2 = \\ &= \int_{\mathbb{R}} (1 + |y|^2)^r \partial_y u \partial_y u dy = - \\ &= - \int_{\mathbb{R}} u \partial_y ((1 + |y|^2)^r \partial_y u) dy = - \\ &= - \int_{\mathbb{R}} 2r(1 + |y|^2)^{r-1} y u \partial_y u dy - \\ &= - \int_{\mathbb{R}} (1 + |y|^2)^r u_{yy} u dy \end{aligned}$$

Since that $y^2 \leq (1 + |y|^2)^2$

It follows from the Cauchy-Schwarz's inequality

$$\begin{aligned} |Bu|_{r,2}^2 &\leq \\ &\leq 2r |M_r \partial_y u|_2 \cdot |M_r u|_2 + |M_r u|_2 |M_r \partial_y^2 u|_2 \end{aligned}$$

On the other hand, an elementary calculation based on the identity

$$\begin{aligned} |\Lambda^2 u|^2 &= |(I - \partial_y^2) u|^2 = |u|^2 + \\ &+ |\partial_y^2 u|^2 - 2u \partial_y^2 u \end{aligned}$$

Shows that

$$|\partial_y^2 u|_{r,0}^2 \leq c |u|_{r,0}^2 + |u|_{r,2}^2$$

And so, by applying the embedding $H_r^2 \subset H_r^0$

It follows that

$$|Bu|_{r,2}^2 = |\partial_y^2 u|_{r,0}^2 \leq c|u|_{r,2}^2 + |u|_{r,2}^2$$

Hence

$$|Bu|_{r,2} \leq c|u|_{r,2}$$

1 Proposition1. (local existence) Given $u_0 \in H_r^2(\mathbb{R})$ there exists $T \in (0, \infty)$ and an unique mild solution u of (1) in $\mathbb{R} \times (0, T)$ with initial data u_0 . Here a mild solution is an element of $C([0, T]; H_r^2(\mathbb{R}))$ satisfying

$$u(t) = e^{Bt}u_0 + \int_0^t e^{B(t-\sigma)}B\left(\frac{u(\sigma)^2}{2}\right)d\sigma$$

We use Lemmal for concluding that B is the infinitesimal generator of a uniformly continuous semigroup of operators on $H_r^2(\mathbb{R})$. The conclusion follows using standard arguments given in Pazy[7].

Proof of Theorem 1

Note that Bf is differentiable, $u(x, t)$ is a classical solution of (1) on any t –interval where it exists. Furthermore

$$\partial_t u = Bu + Bfu \in C([0, T]; H_r^2(\mathbb{R}))$$

For every $s \geq 0$ the existence of a global H^s – valued solution for (1) has already been proved (cf.[1],[2]). In addition such solution belongs to $C^0([0, \infty); H^s(\mathbb{R}))$.

We claim that for every $T > 0$, there exists some constant $M_T > 0$ such that

$$\sup_{0 \leq t \leq T} |u(\cdot, t)|_{r,2} \leq M_T$$

In fact, for every $u_0 \in H_r^2(\mathbb{R})$ by the density of $\mathcal{S}_{2r}^4(\mathbb{R})$ in $H_r^2(\mathbb{R})$ (cf.[14])there exists $(u_{0n}) \in \mathcal{S}_{2r}^4(\mathbb{R})$ such that

$$u_{0n} \rightarrow u_0 \text{ strong in } H_r^2(\mathbb{R})$$

Now consider $u_n = u_n(x, t)$ be the solution of initial value problem.

$$\begin{cases} \partial_t u_n + \partial_x u_n + u \partial_x u_n - \partial_x^2 \partial_t u_n = 0, x \in \mathbb{R}, t > 0 \\ u_n(x, 0) = u_{0n}(x), \quad x \in \mathbb{R} \end{cases}$$

Then the solution u_n exists globally in $C([0, \infty); \mathcal{S}_{2r}^4(\mathbb{R}))$, (cf.[13]). In particular given any $T > 0$ there exists a constant M_T (independent of n) such that

$$|||u_n|||_{2r,4} \leq M_T$$

And (u_n) converges to the solution $u = u(x, t)$ of the problem (2). Since we have the embedding injection $\mathcal{S}_{2r}^4(\mathbb{R}) \subset H_r^2(\mathbb{R})$ it follows that

$$|u_n(\cdot, t)|_{r,2} \leq C |||u_n|||_{2r,4} \leq M_T$$

Finally, we claim that

$$|u(\cdot, t)|_{r,2} \leq \liminf_{n \rightarrow \infty} |u_n(\cdot, t)|_{r,2} \leq M_T$$

This completes the proof.

2.2 LEMMAS FOR THE BBM SYSTEM

Lemma 2.1 If A is as above, and let $\eta > 0, \delta > 0, r \geq 0$ such that

$$0 < a < \min \left\{ 1, \frac{\delta}{\eta \delta + C_r}, \frac{\eta}{\eta \delta + C_r} \right\} \text{ for some } C_r > 0, \text{ then } A \text{ is invertible and bounded below.}$$

Lemma 2.2 Let $g = (g_1, g_2) \in X_r$, and consider the assumptions of Lemma 2.1. Then, the following holds.

- i) $K * \frac{\partial g}{\partial x} \in X_r$, and there exists $C > 0$ such that $\left| K * \frac{\partial g}{\partial x} \right|_{X_r} \leq C |g|_{X_r}$
- ii) $A^{-1}g = K * g$, where $K = (K_{mn})_{1 \leq m, n \leq 2}$ and each of the K_{mn} 's satisfies

$$K_{mn} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ixy} a_{mn}(y) dy, i = \sqrt{-1} \text{ and we set}$$

$$\widehat{A^{-1}g}(y) = (a_{mn}(y)) \hat{g}(y), \hat{g}(y) = \begin{pmatrix} \hat{g}_1(y) \\ \hat{g}_2(y) \end{pmatrix}$$

Theorem 1.1 Let $w_0 = (u_0, v_0) \in X_r$ be as above and suppose the assumptions from Lemma 2.1 hold.

Then there exists a unique solution of the coupled system (1.1) and (1.2) satisfying $w \in$

$$C^0([0, T_0]; H_r^2(\mathbb{R})) \times C^0([0, T_0]; H_r^2(\mathbb{R})), \text{ for some } T_0 > 0. \text{ In addition, we have } w_t = (u_t, v_t) \in C^0([0, T_0]; H_r^2(\mathbb{R})) \times C^0([0, T_0]; H_r^2(\mathbb{R})).$$

3 DISCUSSIONS

The results are interesting when the stability and their corresponding interactions are analyzed in complementary form for the examples indicated in reference [1].

4 CONCLUSIONS

Throughout this work, it has been used an infinitesimal generator of a strongly continuous semigroup (see Pazy, A. [7]). Whereas the local existence of the solutions of a coupled system of Benjamin-Bona-Mahony (BBM) type in a weighted Sobolev space with initial data, it was considered the domain of

the infinitesimal generator as a Banach space equipped with a suitable norm and so, we have defined a mapping Φ which it was proved within the main result, as being a contraction mapping the closed R -ball $(Y_R(T))$ in $X(T) = C^0([0, T]; H_r^2(\mathbb{R})) \times C^0([0, T]; H_r^2(\mathbb{R}))$ into itself. Hence, we found a unique fixed point $w(t)$ for the operator Φ within the ball of radius R . On the other hand, it also was proved that w is in fact differentiable and solve the initial value problema

$$\begin{cases} Aw_t + B w_x + (F w)_x = 0 \\ w(x, 0) = w_0(x) \end{cases}$$

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