Optimization of the mathematical programming and applications

Optimización de la programación matemática y aplicaciones

DOI: 10.46932/sfjdv2n5-114

Received in: Oct 1st, 2021
Accepted in: Dec 30th, 2021

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ABSTRACT
The present investigation responds to the need to solve optimization problems with optimality conditions. The KKT conditions are considered for multiobjective optimization problems with interval-valued objective functions.

Keywords: KKT conditions, interval-valued objective functions.

RESUMEN
La presente investigación responde a la necesidad de resolver problemas de optimización con condiciones de optimalidad. Se consideran las condiciones de KKT para problemas de optimización multiobjetivo con funciones objetivo valoradas en intervalos.

Palabras clave: Condiciones de KKT, funciones objetivo intervalo-valoradas.

1 INTRODUCTION

We are concerned to the existence of an optimal solution for an optimization problem in the case of an interval-multivalued functions using a reformulation about the Karush-Khun-Tucker based on generalized Hukuhara derivative. In order to study the Karush-Khun-Tucker optimality conditions. We handle new relations of partial order within of the class of all closed and bounded intervals in R and we denote it by I , which will be extended to the interval vectors whose components belong to I . Whereas about the arithmetic of elements in I one have used the arithmetic of intervals according to Moore [5].

Ishibuchi and Tanaka [6] considered multiobjective programming problems with interval-valued objective functions and proposed the ordering relation between two closed intervals. The difference of Hukuhara is extended to the generalized difference, so the H-Derivative give rise to the Generalized Derivative justly gH-derivative introduced by Stefanini [7]. Chalco [3] introduce a new relationships in spite of the partial order for reformulate the conditions of The interval-valued function for the optimization problems applying the gH-derivative, taking account the partial order relationship given by Wu [9] . Also, there is an extension to fussy sets due to Stefanini [8] using the gH-derivative.

We are interested in Pareto optimal solution to The interval-multivalued objetive function optimization problems whose range belongs to the set I^n , more precisely for the following mathematical programming

$$F(x) = \left(F_1(x), \dots, F_q(x)\right) \quad (P)$$

$$\text{Subjet to} \qquad g_i(x) \leq 0 \;, \;\; i = 1, \dots, m$$

In such a way the Karush –Khun-Tucker optimality conditions hold for F to be The interval-multivalued function (its components are interval-valued functions) using the generalized gH-derivative. The treatment of the H-derivative and gH-derivative were developed in the papers by Wu [9], Stefanini [7] and Chalco [3]. The study of interval-valued functions allows face up to the solution for the mathematical programming that involve uncertainty without consider the optimization random problem. Since we are treating with interval-valued functions we need to compare the closed and bounded intervals in R.

Gutiérrez,F., J.P. solves the optimization problems analitically and using numerical methods in his research work [4].

2 METHODS AND MATERIALS

Let us denote by I the class of all closed and bounded intervals in R, where it holds a certain arithmetic of intervals.

Let $C = [c^I, c^S]$, $D = [d^I, d^S]$ be in I . then it holds

(i) $C + D = \{c + d \,;\, c \in C \; y \; d \in D\} = [c^I + d^I, c^S + d^S]$.

(ii) For k a real number

$$kC = \{kc; c \in C\} = \{[kc^I, kc^S], \quad if \;\; k \geq 0 \quad [kc^S, kc^I], \;\; if \;\; k < 0$$

Also, one define the Hausdorff´s metric

$$d_H(C,D) = máx\{|c^I - d^I|, |c^S - d^S|\}$$

On the other hand the function $f : R^n \to I$ defined on the Euclidean space R^n is named an interval-valued functions when its range is a closed interval, that is $f(x) = (f_1(x), \dots, f_n(x))$ is closed interval in R, para cada x ∈ R n . The interval-valued function f stand for by $f(x) = [f^I(x), f^S(x)]$, where f^I and f^S are real functions defined on R^n and that hold the condition $f^I(x) \leq f^S(x)$, for all $x \in R^n$.

Let f be the interval-valued function defined on R^n and $A = [a^I, a^S]$ a closed interval in R, whose width is given by $w(A) = a^S - a^I$. Letting $c \in R^n$, we recall that

$$f(x) = A$$

If for each $\varepsilon > 0$, there exists $\delta > 0$ in such a way that $d_H(f(x), A) < \varepsilon$ whenever $\|x - c\| < \delta$, with this in mind and using the Hausdorff´s metric one verifies the following

properties.

Let F be the interval-valued function defined on R^n

$$F(x) = [F^I(x), F^S(x)] \text{ y } A = [a^I, a^S] \in I.$$

1. We have

$$F(x) = A$$

If and only if $F^I(x) = a^I$ and $F^S(x) = a^S$

2. $F(x)$ is continuous in the point c If and only if F^I and F^S are continuous in c
3. F is differentiable in the point c if the real functions F^I and F^S are differentiables in c

On the other hand, there exists another definition of differentiability for the interval-valued function. First and foremost we define the difference of Hukuhara. Let $A = [a^I, a^S]$ and

$B = [b^I, b^S]$ be two elements of I . If there exists a closed interval $C = [c^I, c^S]$ in such a way that $A = B + C$, then C is called the difference of Hukuhara. In this case

$C = [a^I - b^I, a^S - b^S]$ and we denote it by $C = A \ominus B$. Hence, we say that there exists the difference of Hukuhara $C = A \ominus B$ whenever $a^I - b^I \leq a^S - b^S$. Also, if $w(B) < w(A)$ then the difference of Hukuhara makes sense.

Definition 1. Let U be an open set in R. The interval-valued function $f : U \to I$ is called H-differentiable (or differentiable strongly) in the point $x_0 \in R$ if there exists a closed interval $I(x_0) \in I$ (depending on x_0) such that the limits

$$\frac{f\,(x_0+h) \ominus f\,(x_0)}{h} \quad and \frac{f\,(x_0) \ominus f\,(x_0-h)}{h}$$

Boths exists and they are equal to $I(x_0)$. So, $I(x_0)$ is called the H-derivative of f in the point x_0.

The Hukuhara's derivative is very restrictive since the difference of Hukuhara there's not always exists. Because is limited by its own definition. Hence, we define

Definition 2. (Stefanini & Bede [7] We define the difference generalized of Hukuhara that we denote by gH-difference, between two closed interval $C = [c^I, c^S]$ and $D = [d^I, d^S]$, belonging to I

$$C \ominus_g D = E \Leftrightarrow \{C = D + E \;, \quad if\; w(C) \geq w(D)\; D = C + (-1)E \;, \quad if\; w(C) < w(D) \qquad (1)$$

Of course, for all $C \in I$ we have $C \ominus_g C = \{0\} = [0, 0]$, where the unit sets $\{0\}$ in R are considered like degenerate closed interval and they are denoted by [a, a]. Moreover, the gH-difference always exists between two closed interval on R, because for all couple of elements of I are obtained the gH-diference. According to Stefanini & Bede [7] for all closed interval $C = [c^I, c^S]$ and $D = [d^I, d^S]$ in I, one obtain

$$C \ominus_g D = [\{c^I - d^I, c^S - d^S\}, \{c^I - d^I, c^S - d^S\}]$$

Is easy readly to see that for all $A, B, C \in I$ one can have the following

i) $k(A \ominus_g B) = kA \ominus_g kB \;, \; k \in R$

ii) $A \ominus_g B = [0,0]$ if and only if $A = B$

iii) $(A + B) \ominus_g (A + C) = (B \ominus_g C)$

Definition 3. Let U be an open set in R and $t_0 \in U$. The generalized derivative of Hukuhara, which is named by gH-derivative, of an interval-valued function $f : U \to I$ in the point t_0, is defined by

$$f'(t_0) = \frac{f\,(t_0+h) \ominus_g f\,(t_0)}{h} \qquad (2)$$

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If there exists $f'(t_0) \in I$ satifying (2), the we say that f is gH-differentiable in the point t_0. One say that the interval-valued function $f : U \rightarrow I$ is gH−differentiable on U if f is gH-differentiable for each $t_0 \in U$. One result due Chalco[3] yields for $f : U \rightarrow I$ such that $F(t) = [F^I(t), F^S t)]$ if F^I and F^S are differentiable in the point $t_0 \in U$. Then F is gH-differentiable in the point t_0 and verifies

$$F'(t_0) = [\{(F^I)'(t_0), (F^S)'(t_0)\}, \{(F^I)'(t_0), (F^S)'(t_0)\}]$$

On the other hand the converse is not true. That is the gH-differentiability of F not imply in the differentiability of F^I and F^S (in the usual sense).

For instance, if one consider the interval-valued function F, given by $F(t) = [-|kt|, |kt|]$, $k \neq 0$, this implies that F is gH−differentiable in the point $t_0 = 0$ and $F'(0) = [-k, k]$. However F^I and $F^S$are not differentiable in $t_0 = 0$. In addition, for all $t_0 \in U$ and

$F : U \rightarrow I$ to be an interval-valued function such that $F(t) = [F^I(t), F^S t)]$. One can see according to Chalco [3] , F is gH-differentiable in the point $t_0 \in U$ whenever one of the following cases hold

1. F^I and F^S are differentiable in t_0
2. There exists all the one-sided derivatives $(F^I)'_-(t_0)$, $(F^I)'_+(t_0)$, $(F^S)'_-(t_0)$ and $(F^S)'_+(t_0)$ verifying $(F^I)'_-(t_0) = (F^S)'_+(t_0)$ and $(F^I)'_+(t_0) = (F^S)'_-(t_0)$

We have the natural extension in R^n like to be

Definition 4. Let U be an open set in R^n, $F : U \rightarrow I$ an interval-valued function such that $F(t) = [F^I(t), F^S t)]$, $t = (t_1(0), \dots, t_n(0)) \in U$ and the interval-valued function h_i defined by $h_i(t_i) = F(t_1(0), \dots, t_{i-1}(0), t_i(0), t_{i+1}(0), \dots, t_n(0))$. If h_i is gH-differentiable in the point $t_i(0)$, then we say that F has the ith partial gH -derivative in t_0 and we denote by $\left(\frac{\partial F}{\partial t_i}\right)_g (t_0) = h_i{}'(t_i(0))$.

3 OPTIMIZATION FOR INTERVAL-VALUED FUNCTION

We have the following mathematical programming

$$f(x) = [f^I(x), f^S(x)] \qquad (P1)$$

Subjet to $x = (x_1, \dots, x_n) \in U \subset R^n$

$$f(x) = [f^I(x), f^S(x)] \qquad (P2)$$

Subjet to $g_i(x) \leq 0$, $i = 1, \dots, m$

Also, the mathematical programming (P2) can be rewritten like the mathematical programming (P1) considering by $U = \{x \in R^n / g_i(x) \leq 0, i = 1, \ldots, m\} \subset R^n$, the admissible set, where $g_i, i = 1, \ldots, m$ are real constraint functions defined on R^n. In order to solve these mathematical programming, we will give an order relationship on I .

4 DISCUSSION

Let U be an convex set on R^n and consider the interval-multivalued function

$F(x) = \left(F_1(x), \ldots, F_q(x)\right)$ Where we recall that

$F_i(x) = [F_i^I(x), F_i^S(x)]$, $x \in U$ for $i = 1,2, \ldots, q$ are all to be interval -valued functions

$$F(x) = \left(F_1(x), \ldots, F_q(x)\right) \quad (MP1)$$

subject to $x = (x_1, \ldots, x_n) \in U \subset R^n$

$$F(x) = \left(F_1(x), \ldots, F_q(x)\right) \quad (MP2)$$

subject to $g_i(x) \leq 0$, $i = 1, \ldots, m$

Where the constraint functions $g_i : R^n \to R$ are convex functions on R^n, for all , $i = 1, \ldots, m$. On the other hand we can rewrite the problem (MP2) considering the admissible set $U = \{x \in R^n ; g_i(x) \leq 0 , i = 1, \ldots, m\} \subset R^n$

Definition 5.- Let $C = [c^I, c^S]$, $D = [d^I, d^S]$ be two elements of I. We have

$$C \prec_{IS} D \quad \text{when and only when} \quad c^I \leq d^I \text{ and } c^S \leq d^S$$

Of course $\prec_{IS}$ is a partial order relation on I because is reflexive, antisymmetric and transitive.

We give some definitions of optimal solution, called a Pareto optimal solution.

Definition 6.- Let x^* be a feasible solution to the problem (MP1) we say that x^*

(i) Is a Pareto optimal solution of the type-I (resp. type- II and type-III) for the problem (MP1) if there is no exists $x \in U$ such that

$$F(x) \prec_{IS} F(x^*), (Resp. F(x) \prec_{CR} F(x^*) \text{ and } F(x) \prec_{IW} F(x^*))$$

(ii) Is a strongly Pareto optimal solution of the type-I (resp. type-II and type-III)for the problem (MP1) if there is no exists $x \in U$ such that

$$F(x) \precsim_{IS} F(x^*), (Resp. F(x) \prec_{CR} F(x^*) \text{ and } F(x) \prec_{IW} F(x^*))$$

(iii) Is a weakly Pareto optimal solution of the type-I (resp. type- II and type-III)for the problem (MP1) if there is no exists $x \in U$ such that

$F_i(x) \prec_{IS} F_i(x^*), (Resp. F_i(x) \prec_{CR} F_i(x^*) \text{ and } F_i(x) \prec_{IW} F_i(x^*))$ for all $i = 1, \dots, q$. Now we denote $U_p^{(I)}$, $U_{sp}^{(I)}$ and $U_{wp}^{(I)}$ to be the sets of all solutions according to the above definition given in (i),(ii) and (iii) respectively, of course $U_{sp}^{(i)} \subset U_p^{(i)} \subset U_{wp}^{(i)}$, where i can be of the type I , II or III, is straight forward the following properties:

1. $U_{sp}^{(I)} \subset U_{sp}^{(III)}$
2. $U_p^{(I)} \subset U_p^{(III)}$
3. $U_{Wp}^{(I)} \subset U_{Wp}^{(III)}$

Theorem 1.- Let (MP2) be the interval-valued programming problem, $x^* \in U$. Suppose that F is continuously gH-differentiable in the point x^* ; F_j^I and F_j^S are convex functions for all $j = 1,\dots,q$. If there exists the Lagrange´s multipliers

$0 < \lambda_j \in R$, $j = 1, \dots, q$ and $0 \leq \mu_j \in R$, $j = 1, \dots, m$ in such a way that KKT hold

i) $\sum_{j=1}^{q} \lambda_j \nabla(F_j^I + F_j^S)(x^*) + \sum_{j=1}^{m} \mu_j \nabla g_j(x^*) = 0$

ii) $\mu_j g_j(x^*) = 0$ for all $j = 1, \dots, m$. Then $x^* \in U_P^{(I)}$ and $x^* \in U_P^{(III)}$ for the (MP2) mathematical programming.

A direct consequence of the theorem Karush-khun-Tucker to real functions, one have the following result.

Theorem 2.- Let (MP2) be the mathematical programming and consider F to be IW-convex and (weak) differentiable continuously in the point x^* . If there exists the Lagrange´s multipliers

$0 < \lambda_j^I, \lambda_j^W \in R$, $j = 1, \dots, q$ and $0 \leq \mu_j \in R$, $j = 1, \dots, m$ in such a way that KKT hold

iii) $\sum_{j=1}^{q} \lambda_j^I \nabla F_j^I(x^*) + \sum_{j=1}^{q} \lambda_j^W \nabla F_j^W(x^*) + \sum_{j=1}^{m} \mu_j \nabla g_j(x^*) = 0$

iv) $\mu_j g_j(x^*) = 0$ for all $j = 1, \dots, m$. Then $x^* \in U_P^{(III)}$ for the (MP2) mathematical programming.

In the sequel we give an application of the above theorem. Let F be an interval multivalued function

$F(x_1,x_2) = (F_1(x_1,x_2), F_2(x_1,x_2))$ whose components are given by

$$F_1(x_1,x_2) = [x_1^2 + 2x_1 + x_2^2 - 2x_2 + 3, x_1^2 + 2x_1 + x_2^2 - 2x_2 + 4]$$

$$F_2(x_1,x_2) = [2x_1^2 + 4x_1 + 2x_2^2 - 4x_2 + 7, 2x_1^2 + 4x_1 + 2x_2^2 - 4x_2 + 7]$$

We have the following mathematical programming (type MP2)

$$min\ F(x_1,x_2) = (F_1(x_1,x_2), F_2(x_1,x_2))$$

$$subject\ to \quad -x_1 - x_2 + 1 \le 0 \ ,$$

$$-3x_1 - x_2 + 4 \le 0 \quad , \qquad -x_1 - 1 \le 0 ,$$

$$-x_2 + 1 \le 0$$

Then we have the interval -valued function F, with its component respectives

$$F_1^I(x) = x_1^2 + 2x_1 + x_2^2 - 2x_2 + 3 \quad \Rightarrow F_1^W(x) = 1$$

$$F_2^I(x) = 2x_1^2 + 4x_1 + 2x_2^2 - 4x_2 + 7 \qquad \Rightarrow F_2^W(x) = 1$$

And the constraint functions: $g_1(x) = -x_1 - x_2 + 1$, $g_2(x) = -3x_1 - x_2 + 4$,

$g_3(x) = -x_1 - 1, g_4(x) = -x_2 + 1$. Therefore the Karush–Kuhn–Tucker (KKT) conditions imply in the following system

$$\lambda_1^I(2x_1 + 2) + \lambda_2^I(4x_1 + 4) - \mu_1 - 3\mu_2 - \mu_3 = 0 \quad ,$$
$$\lambda_1^I(2x_2 - 2) + \lambda_2^I(4x_2 - 4) - \mu_1 - \mu_2 - \mu_4 = 0$$

Hence, one obtain $(x_1^*, x_2^*) = (4/5, 8/5)$, $\lambda_1^I = \frac{1}{2}$, $\lambda_2^I = \frac{1}{4}$, $\mu_1 = \mu_3 = \mu_4 = 0$ and $\mu_2 = \frac{6}{5}$.

On the other hand $\mu_i g_i(x^*) = \mu_i g_i(9/5, 3/5) = 0$ for all $i = 1, \dots, 4$. Then we deduce that $(x_1^*, x_2^*) = (4/5, 8/5)$ is a Pareto optimal solution of type -II

The information obtained through the KKT conditions can be implemented by numerical methods according to [4].

5 CONCLUSIONS

1. The analysis result gave rise a new knowledge about the methodology for obtaining solutions of the interval-multivalued programming problem using the KKT conditions.

2. three types of partial order relationship was established on I . These relationship allowed perform the comparisons between the vectors whose components are of interval-valued (belonging to I)

3. According to the formulation about the gH-derivative was reformulated The Karush Khun Tucker conditions for the mathematical programming whose objective functions are the interval-multivalued functions (its components are of interval-valued functions)

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